Problem 3: Momentum and Angular Momentum in Field

From Maxwell's Equation

$$\vec{P} = \int d^3\vec{x} \, \frac{\vec{E}(\vec{x}) \times \vec{B}(\vec{x})}{4\pi c} = \vec{p}_{(\vec{x})} \text{ momentum density}$$

$$\vec{J} = \int d^3\vec{x} \left(\vec{x} \times \vec{p}_{(\vec{x})} \right)$$

Quantized field $\hat{\vec{A}}(\vec{x}) = \hat{\vec{A}}^{(+)}_{(\vec{x})} + \hat{\vec{A}}^{(-)}_{(\vec{x})}$

$$\hat{\vec{A}}^{(+)}(\vec{x}) = \sum_{\vec{k},\lambda}' \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \, \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{A}}^{(-)} = \left(\hat{\vec{A}}^{(+)}\right)^{\dagger}$$

$$\hat{\vec{E}}^{(+)} = i\sum_{\vec{k},\lambda}' \sqrt{\frac{2\pi\hbar\omega_k}{V}} \, \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{B}}^{(+)} = i\sum_{\vec{k},\lambda}' \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left(\vec{e}_k \times \vec{\epsilon}_{\vec{k},\lambda}\right) e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

Notes: $$\int_V d^3x \, \frac{e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}{V} = \delta_{\vec{k},\vec{k}'}$$

$$\vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'}$$

(3a) Plug mode decomposition into $\vec{\hat{P}}$

$$\Rightarrow \quad \vec{\hat{P}} = \int d^3x \left(\frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} + \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(+)}}{4\pi c} + h.c. \right)$$

Consider first term:

$$\int d^3x \; \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} = \sum_{\vec{k},\vec{k}',\lambda,\lambda'} \frac{1}{4\pi c} \left(2\pi\hbar\sqrt{\omega_k \omega_{k'}}\right) \vec{\epsilon}_{\vec{k},\lambda} \times \left(\vec{e}_{\vec{k}'} \times \vec{\epsilon}^{\,*}_{\vec{k}',\lambda'}\right) \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\delta_{\vec{k},\vec{k}'}} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k}',\lambda'}$$

$$= \sum_{\vec{k}} \sum_{\lambda,\lambda'} \frac{\hbar\omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times \left(\vec{e}_{\vec{k}} \times \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}\right)\right]}_{\vec{e}_{\vec{k}} \left(\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}\right) - \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} \left(\vec{e}_{\vec{k}} \cdot \vec{\epsilon}_{\vec{k},\lambda}\right)} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda'}$$

$$\left(\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}\right) = \delta_{\lambda\lambda'}, \qquad \left(\vec{e}_{\vec{k}} \cdot \vec{\epsilon}_{\vec{k},\lambda}\right) = 0$$

$$= \sum_{\vec{k}} \frac{\hbar \vec{k}}{2} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda}$$

having used $\vec{k} = \frac{\omega}{c} \vec{e}_{\vec{k}}$

by similar steps, using $\int \frac{d^3x}{V} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta_{\vec{k}',-\vec{k}}$

$$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} = \sum_{\vec{k}}' \sum_{\lambda\lambda'}' \frac{\hbar\omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times (\vec{e}_{-\vec{k}} \times \vec{\epsilon}_{-\vec{k},\lambda'})\right]}_{\vec{e}_{-\vec{k}}\,(\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'})} \hat{a}_{\vec{k},\lambda} \hat{a}_{-\vec{k},\lambda'}$$

<u>Aside</u>: $\vec{e}_{-\vec{k}} = -\vec{e}_{\vec{k}}$, thus by symmetry, when we sum over all $\vec{k}$,

$$\sum_{\lambda\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'}) \xrightarrow[\vec{k}\to-\vec{k}]{} \sum_{\lambda\lambda'} \vec{e}_{\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'})$$

$$= -\sum_{\lambda,\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda'} \cdot \vec{\epsilon}_{\vec{k},\lambda})$$

Thus the terms cancel pairwise.

$$\Rightarrow \hat{\vec{P}} = \sum_{\vec{k},\lambda}' \frac{\hbar\vec{k}}{2} (\hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} + h.c.)$$

$$= \sum_{\vec{k},\lambda} \hbar\vec{k} (\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} + \frac{1}{2})$$

But $\sum_{\vec{k}} \frac{\hbar\vec{k}}{2} = 0$ (vector's cancel)

$$\Rightarrow \boxed{\hat{\vec{P}} = \sum_{\vec{k},\lambda} \hbar\vec{k} \, (\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda})}$$

Neat! Momentum $= \hbar\vec{k} \times$ number of photons

(1b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \;\; \vec{x} \times \hat{\vec{\mathcal{P}}}(\vec{x})$$

where $\hat{\vec{\mathcal{P}}}(\vec{x}) = \frac{1}{4\pi c}\,(\hat{\vec{E}} \times \hat{\vec{B}})$ = momentum density

Lets massage these equations a bit.

$$(\hat{\vec{E}} \times \hat{\vec{B}})_i = \epsilon_{ijk}\, E_j\, B_k \quad \text{(summation convention)}$$

$$= \epsilon_{ijk}\, E_j\, \epsilon_{k\ell m}\, \partial_\ell A_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\, E_j \partial_\ell A_m$$

$$= E_\ell\, \partial_i A_\ell - E_\ell \partial_\ell A_i$$

Now $\vec{J} = \int d^3x \;\; \vec{x} \times \vec{\mathcal{P}}(\vec{x})$

$$\Rightarrow \; J_j = \epsilon_{jki} \int d^3x \;\; x_k\, \mathcal{P}_i$$

$$= \epsilon_{jki}\, \frac{1}{4\pi c} \int d^3x \left[E_\ell\,(x_k \partial_i) A_\ell - (x_k E_\ell)(\partial_\ell A_i) \right]$$

$$= \frac{1}{4\pi c} \int d^3x \; E_\ell\, (\vec{x} \times \vec{\nabla})_j\, A_\ell$$

$$+ \frac{1}{4\pi c} \int d^3x\, \epsilon_{jki}\, \underbrace{\partial_\ell (x_k E_\ell)}_{\left[\delta_{\ell k} + \vec{\nabla}\cdot\vec{E}\right]}\, A_i \qquad \text{(integration by parts)}$$

$\vec{\nabla}\cdot\vec{E} \to 0$ in free space

$$\Rightarrow J_j = \frac{1}{4\pi c}\left(\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla})_j \, A_\ell \right.$$
$$\left. + \int d^3x \; (\vec{E}\times\vec{A})_j \right)$$

$$\Rightarrow \vec{J} = \vec{J}_{orb} + \vec{J}_{spin}$$

$$\boxed{\begin{aligned} \vec{J}_{orb} &= \frac{1}{4\pi c}\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla}) \, A_\ell \\ \vec{J}_{spin} &= \frac{1}{4\pi c}\int d^3x \; (\vec{E}\times\vec{A}) \end{aligned}}$$

(1c)

Let us expand these terms in the plane wave basis:

$$\vec{J}_{orb} = \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)} \quad + h.c. \right)$$
$$+ \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(+)} (\vec{x}\times\vec{\nabla}) \vec{A}_\ell^{(+)} \quad + h.c. \right)$$

Consider

$$\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}} \; \hat{a}^{\dagger}_{\vec{k}\lambda} \hat{a}_{\vec{k}',\lambda'} \; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'} \qquad \left(\begin{matrix}\text{Summing} \\ \text{over } \ell\end{matrix}\right)$$

$$\underbrace{\int \frac{d^3x}{V} \, e^{-i\vec{k}\cdot\vec{x}} \, (\vec{x}\times -i\vec{\nabla}) \, e^{i\vec{k}'\cdot\vec{x}}}_{\mathcal{L}}$$

Aside: $\int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x}\times -i\vec{\nabla}\right) e^{i\vec{k}'\cdot\vec{x}}$

$$= \int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x}\times \vec{k}'\right) e^{i\vec{k}'\cdot\vec{x}}$$

$$= \left[\int \frac{d^3x}{V}\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\ \vec{x}\right] \times \vec{k}'$$

$$= -i\vec{\nabla}_{\vec{k}'} \underbrace{\left[\int \frac{d^3x}{V}\, e^{-i(\vec{k}-\vec{k}')}\right]}_{\delta_{\vec{k},\vec{k}'}} \times \vec{k}'$$

$$\Rightarrow \frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)} \left(\vec{x}\times\vec{\nabla}\right) A_\ell^{(-)}$$

$$= \sum_{\vec{k},\vec{k}'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\ \hat{a}_i(\vec{k})\ \hat{a}_i^\dagger(\vec{k}') \left(-i\left(\vec{\nabla}_{k'}\ \delta(\vec{k}-\vec{k}')\right)\times\vec{k}'\right)$$

~~Integration~~ where $\hat{\vec{a}}(\vec{k}) \equiv \sum_\lambda \vec{\epsilon}_\lambda(\vec{k})\ \hat{a}_\lambda(\vec{k})$

Integration by parts

$$\Rightarrow \frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)} \left(\vec{x}\times\vec{\nabla}\right) A_\ell^{(+)}$$

$$= \frac{\hbar}{2}\sum_k \hat{a}_i(\vec{k}) \underbrace{\left[+i\vec{k}\times\vec{\nabla}_{\vec{k}}\right]}_{\Uparrow} \hat{a}_i^\dagger$$

k-space Orbital angular momentum operator

Consider

$$\int d^3x \; \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{-i\hbar}{2} \sum_{\vec{k}\vec{k}',\lambda\lambda'} \hat{a}^{\dagger}_{\vec{k}\lambda} \hat{a}_{\vec{k}'\lambda'} \; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \times \vec{\epsilon}_{\vec{k}',\lambda'} \; \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\to\ \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \frac{-i\hbar}{2} \sum_{\vec{k}} \left[\left(\vec{\epsilon}^{\,*}_{\vec{k},+} \times \vec{\epsilon}_{\vec{k},+}\right) \hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} + \left(\vec{\epsilon}^{\,*}_{\vec{k},-} \times \vec{\epsilon}_{\vec{k},-}\right) \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right]$$

Aside $\vec{\epsilon}_{\vec{k},\pm} = \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ where $\vec{e}_1$ and $\vec{e}_2$ are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{k}$

$$\Rightarrow \vec{\epsilon}^{\,*}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = \pm i \, \vec{e}_k$$

$$\Rightarrow \int d^3x \; \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-} \hat{a}_{\vec{k},-} \right) \vec{e}_{\vec{k}}$$

Now $$\int d^3x \; \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} \left(\hat{a}_{\vec{k}+} \hat{a}^{\dagger}_{\vec{k},+} - \hat{a}_{\vec{k}-} \hat{a}^{\dagger}_{\vec{k}-} \right) \vec{e}_k$$

$$= \frac{\hbar}{2} \sum_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} - \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right) \vec{e}_k$$ (commutator cancels)

Finally note: $\vec{\epsilon}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = 0$

$$\Rightarrow \int d^3x \; \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x \; \vec{E}^{(-)} \times \vec{A}^{(-)}$$

$$= 0$$

Thus
$$\boxed{\vec{J}_{spin} = \hbar \sum_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-} \hat{a}_{\vec{k},-} \right) \vec{e}_k}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carrying~~ carry one hbar of angular momentum along (opposite to) the direction of propagation $\vec{e}_k$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three $(2S+1 = 3)$. This is a very subtle point coming from the fact the the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

Define $\hat{\vec{J}}_{spin} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$

where $\hat{J}_x = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_y = \frac{\hbar}{2i}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_z = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\hat{J}_k$

Check: $[\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4i}\left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-]\right)$

$= \frac{\hbar^2}{4i}\left[2\hat{a}_+^\dagger \hat{a}_+ (\underbrace{[\hat{a}_-^\dagger, \hat{a}_-]}_{-1}) - 2\hat{a}_-^\dagger \hat{a}_- (\underbrace{[\hat{a}_+^\dagger, \hat{a}_+]}_{=-1})\right]$

$= i\hbar\left(\frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)\right) = i\hbar \hat{J}_z$ ✓

$[\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{4}\left([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-]\right.$
$\left. + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\right)$

$= \frac{\hbar^2}{4}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) + \hat{a}_-^\dagger \hat{a}_+(1) - \hat{a}_-^\dagger \hat{a}_+(-1)\right)$

$= \frac{-\hbar^2}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar \hat{J}_y$ ✓

$$[\hat{J}_y, \hat{J}_z] = \frac{\hbar^2}{4i}\Big([\hat{a}_+^\dagger\hat{a}_-, \hat{a}_+^\dagger a_+] - [\hat{a}_+^\dagger\hat{a}_-, \hat{a}_-^\dagger \hat{a}_-]$$
$$- [\hat{a}_-^\dagger\hat{a}_+, \hat{a}_+^\dagger\hat{a}_+] + [\hat{a}_-^\dagger\hat{a}_+, \hat{a}_-^\dagger\hat{a}_-]\Big)$$

$$= \frac{\hbar^2}{4i}\left(\hat{a}_+^\dagger\hat{a}_-(-1) - \hat{a}_+^\dagger\hat{a}_-(1) - \hat{a}_-^\dagger\hat{a}_+(1) + \hat{a}_-^\dagger\hat{a}_+(-1)\right)$$

$$= -\frac{\hbar^2}{2i}\left(\hat{a}_+^\dagger\hat{a}_- + \hat{a}_-^\dagger\hat{a}_+\right)$$

$$= i\hbar\left[\frac{\hbar}{2}(\hat{a}_+^\dagger\hat{a}_- + \hat{a}_-^\dagger\hat{a}_+)\right] = i\hbar\hat{J}_x \quad \checkmark$$

The Schwinger representation is the "second quantized form" of the spin 1/2 operators

$$\hat{J}_x = \frac{\hbar}{2}\left(|+_z\rangle\langle -_z| + |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_y = \frac{\hbar}{2i}\left(|+_z\rangle\langle -_z| - |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_z = \frac{\hbar}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right)$$

"Second quantize" $|\pm_z\rangle \Rightarrow \hat{a}_\pm^\dagger$ create spin up or down

$\langle \pm_z| \Rightarrow \hat{a}_\pm$ annihilate spin up or down

thus, we can easily map the spin angular momentum of the ~~th~~ photon onto the Bloch sphere, also known as the Poincaré sphere as we visited in PS# 1